Recent developments in statistical mechanical interpretation of algorithmic information theory

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ALC2013, September 20th, 2013, Guangzhou, China

Abstract

Algorithmic Information Theory (AIT, for short) is a theory of program-size.

In this talk, we introduce the statistical mechanical interpretation into AIT, and explain some developments of the interpretation by pursuing its formal correspondence to normal statistical mechanics in self-contained manner.

Algorithmic Information Theory (AIT)

AIT: Prefix-free Sets

- $\{0,1\}^* := \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \dots\}$

  The set of finite binary strings

- For any $s \in \{0,1\}^*$, $|s|$ denotes the <u>*length*</u> of s.

- Let $V \subset \{0,1\}^*$. We say V is <u>*prefix-free*</u> if for any distinct s and $t \in V$, s is not a prefix of t.

For example $\{0, 10\}$: prefix-free

$\{0, 01\}$: not prefix-free

AIT: Program-size Complexity

Suppose that $M\colon \{0,1\}^* \to \{0,1\}^*$ is a partial recursive function such that Dom M is a prefix-free set, where Dom M is the domain of definition of M.

Such a partial recursive function M is called a *prefix-free machine*.

$$H_M(s) := \min\left\{ |p| \;\middle|\; p \in \{0,1\}^* \;\&\; M(p) = s \right\} \text{ for each } s \in \{0,1\}^*.$$

Theorem [existence of prefix-free machine]

There exists a prefix-free machine U such that, for each prefix-free machine M, there exists $d \in \mathbb{N}$ with the property that, for every $s \in \{0,1\}^*$,

$$H_U(s) \le H_M(s) + d.$$

Such a prefix-free machine U is called an *optimal prefix-free machine*.

We choose a particular optimal prefix-free machine U as a standard one. Then the *program-size complexity* (or *Kolmogorov complexity*) $H(s)$ of $s \in \{0,1\}^*$ is defined by $H(s) := H_U(s)$.

Quantum Mechanics

Basic Ingredients in Quantum Mechanics

• A quantum system $\mathcal{Q}$ is represented by a finite or infinite dimensional complex Hilbert space $\mathcal{H}$, called a *state space*.

• A state of the quantum system $\mathcal{Q}$ is represented by a vector in the state space $\mathcal{H}$, called a *state vector*.

• The physical quantities of the quantum system $\mathcal{Q}$, such as the coordinates and the components of momentum and the energy of the system, are represented by Hermitian operators on the state space $\mathcal{H}$ whose all eigenvectors form a basis for $\mathcal{H}$, and are called *observables*.

Postulate [Quantum Measurements]

(i) The set of possible outcomes of a measurement of an observable $\hat{A}$ of a quantum system $\mathcal{Q}$ is the eigenvalue spectrum of $\hat{A}$.

(ii) Suppose that a measurement of the observable $\hat{A}$ is performed upon the quantum system $\mathcal{Q}$ in the quantum state represented by a state vector $\Psi \in \mathcal{H}$ which is an eigenvector of A belonging to the eigenvalue m. Then the outcome m is obtained with certainty. □

Basic Ingredients in Quantum Mechanics

• The observable $\widehat{H}$ which represents the energy of a quantum system $\mathcal{Q}$ is called *Hamiltonian*, and the quantum states represented by eigenvectors of $\widehat{H}$ are called *energy eigenstates*. The eigenvalue of $\widehat{H}$ represents the energy of the quantum system $\mathcal{Q}$.

• In statistical mechanics, among all quantum states, *energy eigenstates* are of particular importance.

• Suppose that a measurement of the energy is performed upon the quantum system $\mathcal{Q}$ in the energy eigenstate represented by an eigenvector Ψ_E of $\widehat{H}$ belonging to the energy E. Then, by the postulate, the outcome E is obtained with certainty.

• Thus, if a quantum system $\mathcal{Q}$ is in the energy eigenstate, then the quantum system $\mathcal{Q}$ has a definite energy.

Basic Ingredients in Quantum Mechanics

• Consider two quantum systems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ whose state spaces are $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

• Then the composition $\mathcal{Q}$ of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is also quantum system whose state space is represented by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The quantum system $\mathcal{Q}$ is called the *composite system* (or *total system*) of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and each of the quantum systems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is called a *subsystem* of $\mathcal{Q}$.

• If the quantum system $\mathcal{Q}_1$ is in the state represented by a vector $\Psi_1 \in \mathcal{H}_1$ and the quantum system $\mathcal{Q}_2$ is in the state represented by a vector $\Psi_2 \in \mathcal{H}_2$, then the state of the composite quantum system $\mathcal{Q}$ is represented by the tensor product $\Psi_1 \otimes \Psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ of the vector Ψ_1 and the vector Ψ_2.

Basic Ingredients in Quantum Mechanics

• Let $\hat{H}_1$ and $\hat{H}_2$ be the Hamiltonians of the two quantum systems $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively.

• Then the Hamiltonian of the composite quantum system $\mathcal{Q}$ of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is given by

$$\hat{H}_1 \otimes I_2 + I_1 \otimes \hat{H}_2,$$

where I_1 and I_2 are the identity operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. (We here ignore the interaction between $\mathcal{Q}_1$ and $\mathcal{Q}_2$.)

• Suppose that the quantum system $\mathcal{Q}_1$ is in the energy eigenstate represented by an eigenvector Ψ_1 of $\hat{H}_1$ belonging to the energy E_1 and the quantum system $\mathcal{Q}_2$ is in the energy eigenstate represented by an eigenvector Ψ_2 of $\hat{H}_2$ belonging to the energy E_2. Then the composite system $\mathcal{Q}$ is in the state represented by $\Psi_1 \otimes \Psi_2$ which is an eigenvector of $\hat{H}_1 \otimes I_2 + I_1 \otimes \hat{H}_2$ belonging to the eigenvalue $E_1 + E_2$.

• Thus, if the quantum system $\mathcal{Q}_1$ has a definite energy E_1 and the quantum system $\mathcal{Q}_2$ has a definite energy E_2, then the composite quantum system $\mathcal{Q}$ has a definite energy $E_1 + E_2$. This shows the fact that energy has additivity.

Statistical Mechanical Interpretation of AIT

Intuition

Statistical Mechanical Interpretation of AIT: Intuition

• In the statistical mechanical interpretation of AIT, we consider the correspondence between a prefix-free machine and a quantum system. To be specific, the domain of definition of a prefix-free machine M corresponds to the complete set of energy eigenstates of a quantum system $\mathcal{Q}$, where each program $p \in \operatorname{Dom} M$ corresponds an energy eigenstate of $\mathcal{Q}$.

• For any program $p \in \operatorname{Dom} M$, the program-size $|p|$ corresponds to the energy of energy eigenstate to which the program p corresponds.

• Let M_1 and M_2 be two prefix-free machines. Suppose that M_1 and M_2 correspond to quantum system $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with the state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consider programs $p_1 \in \operatorname{Dom} M_1$ and $p_2 \in \operatorname{Dom} M_2$ which corresponds state vectors $\Psi_1 \in \mathcal{H}_1$ and $\Psi_2 \in \mathcal{H}_2$, respectively. Then the concatenation $p_1 p_2$ corresponds to the tensor product $\Psi_1 \otimes \Psi_2$.

• Note that $|p_1 p_2| = |p_1| + |p_2|$. This equation is interpreted as that the "energy" of $p_1 p_2$ equals to the sum of the "energy" of p_1 and the "energy" of p_2. This is consistent with the additivity of the energy in quantum mechanics.

Statistical Mechanical Interpretation of AIT: Intuition

Example [interpretation of prefix-free machines as quantum systems]

(i) Two level system (i.e. Qubit):

Let B be a prefix-free machine for which $\operatorname{Dom} B = \{1, 01\}$. The corresponding quantum system is a qubit, and has two energy eigenstates where one has energy 1 and the other has energy 2.

(ii) One dimensional harmonic oscillator:

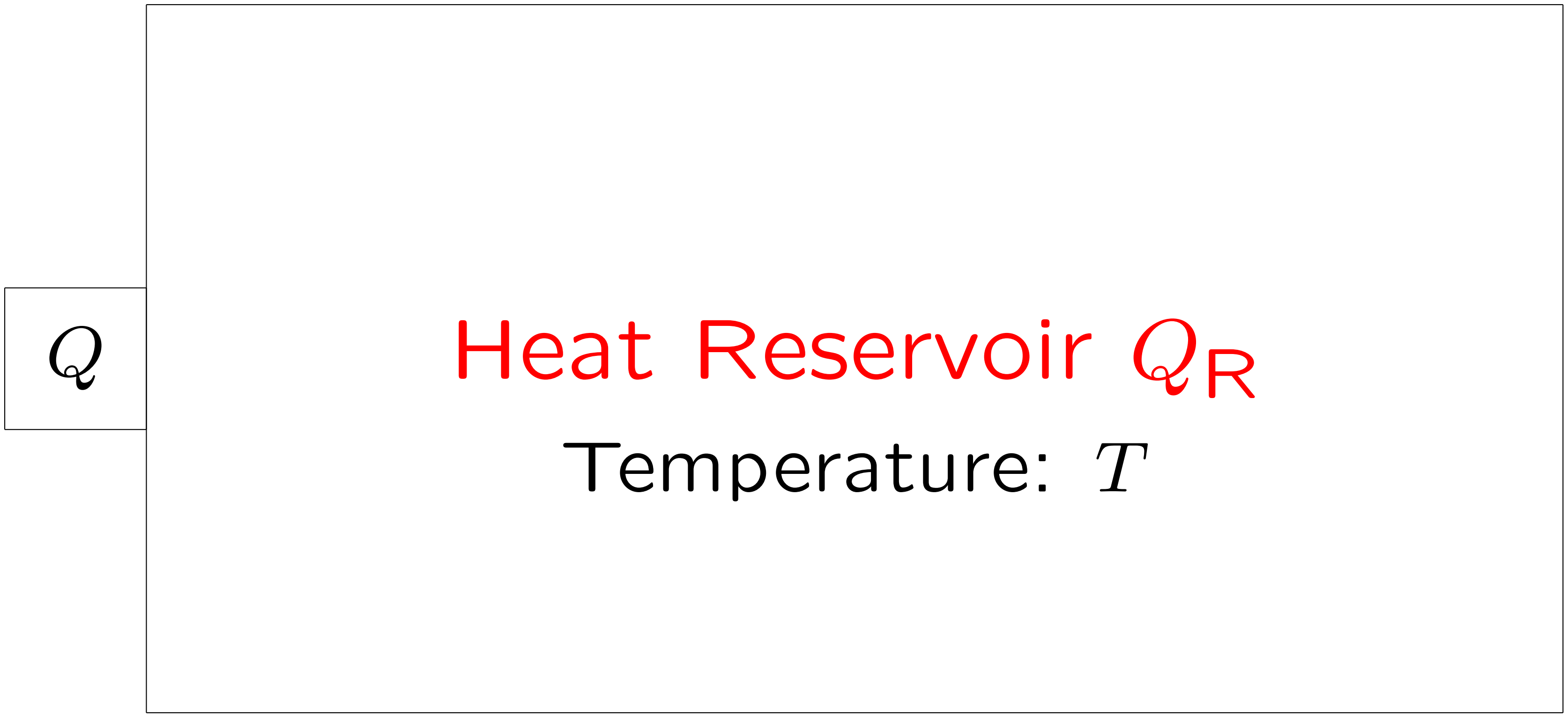
Let O be a prefix-free machine for which $\operatorname{Dom} O = \{0^l 1 \mid l \in \mathbb{N}\}$. The corresponding quantum system is a quantum dimensional harmonic oscillator, and has countably infinite energy eigenstates and the energy eigenvalue spectrum of $\mathbb{N}^+$. The nth lowest energy eigenstate has energy n.

Statistical Mechanics

Statistical Mechanics

Consider a quantum system Q at constant temperature T.

Namely, consider a quantum system Q in thermal contact with a very large quantum system Q_{R}, called *heat reservoir*, whose temperature is T.



Let Q_{total} be the total (composite) quantum system consisting of Q and Q_{R}.

$\boxed{\text{Statistical Mechanics}}$

• In statistical mechanics, among all quantum states, *energy eigenstates* are of particular importance.

• Normally, the Hamiltonian of a quantum system considered in statistical mechanics has discrete eigenvalue spectrum. Thus, an energy eigenstate of the quantum system can be specified by a number $n = 1, 2, 3, \ldots,$ called a *quantum number.*

We identify a quantum number with the corresponding energy eigenstate.

Statistical Mechanics

Definition [(Statistical Mechanical) Entropy]

The *entropy* $S(E)$ of a quantum system with energy E is defined by

$$S(E) := k \ln \Theta(E).$$

Here, $\Theta(E)$ is the number of energy eigenstates whose energy equals to E. The proportional constant k is called *the Boltzmann constant*. $\square$

Definition [Temperature]

The *temperature* T of a quantum system with energy E is defined by

$$\frac{1}{T} = \frac{\partial S}{\partial E}(E).$$

$\square$

Note that the above definitions apply to each of the quantum systems Q, Q_{R}, and Q_{total}.

Statistical Mechanics

The fundamental postulate of statistical mechanics is stated as follows for the total quantum system Q_{total}:

The Principle of Equal Probability

If the energy of the quantum system Q_{total} is known to have a constant value E. then the quantum system Q_{total} is equally likely to be in any energy eigenstate whose energy equals to E. □

As a result, we can show the following for the quantum system Q (and not for Q_{total}):

Result of the Principle of Equal Probability

The probability Prob(n) that the quantum system Q is in an energy eigenstate n with energy E_n is given as:

$$\text{Prob}(n) = \frac{1}{Z} e^{-\frac{E_n}{kT}}.$$

Here, the normalization factor $Z := \sum_n e^{-\frac{E_n}{kT}}$ is called the *partition function* of the quantum system. The distribution Prob(n) is called *the canonical distribution*. □

Statistical Mechanics

Thermodynamic Quantities of the quantum system Q at temperature T

- **Energy** $\quad E = \sum_n E_n \, \mathrm{Prob}(n) = \frac{1}{Z} \sum_n E_n e^{-\frac{E_n}{kT}} = kT^2 \frac{d}{dT} \ln Z.$

The energy E of the quantum system Q is the expected value of an energy E_n of an energy eigenstate n of the quantum system Q at temperature T.

- **Free Energy** $\quad F = -kT \ln Z.$

The free energy F of the quantum system Q is related to the work performed by the system during a process at constant temperature T.

- **(Statistical Mechanical) Entropy** $\quad S = \frac{E - F}{T}.$

Note that the entropy S of the system Q equals to the Shannon entropy of the probability distribution $\{\mathrm{Prob}(n)\}$, i.e., $S = -k \sum_n \mathrm{Prob}(n) \ln \mathrm{Prob}(n)$.

Statistical Mechanical Interpretation of AIT

$\boxed{\text{Introduction of Thermodynamic Quantities into AIT}}$

We introduce the notion of thermodynamic quantities such as free energy, energy, (statistical mechanical) entropy, and specific heat into AIT by performing the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T obeying the canonical distribution.

$\boxed{\text{Replacements}}$ [Calude & Stay 2006] Let M be a prefix-free machine.

$$\begin{array}{lcl} \text{An energy eigenstate } n & \Longrightarrow & \text{A program } p \in \operatorname{Dom} M, \\ \text{The energy } E_n \text{ of } n & \Longrightarrow & \text{The length } |p| \text{ of } p, \\ \text{Boltzmann constant } k & \Longrightarrow & 1/\ln 2. \end{array}$$

□

Immediate Application of the Replacements: Transient Definitions

Perform the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T. (M: prefix-free machine)

An energy eigenstate n	$\Longrightarrow$	A program $p \in \mathrm{Dom}\, M$,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p\rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$. Boltzmann factor: $2^{-\frac{\lvert p\rvert}{T}}$

Partition function $Z(T)=\sum_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad Z_M(T)=\sum_{p\in \mathrm{Dom}\, M} 2^{-\frac{\lvert p\rvert}{T}}$,

Free energy $F(T)=-kT\ln Z(T) \quad \Longrightarrow \quad F_M(T)=-T\log_2 Z_M(T)$,

Energy $E(T)=\frac{1}{Z(T)}\sum_n E_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad E_M(T)=\frac{1}{Z_M(T)}\sum_{p\in \mathrm{Dom}\, M} \lvert p\rvert\, 2^{-\frac{\lvert p\rvert}{T}}$,

Entropy $S(T)=\frac{E(T)-F(T)}{T} \quad \Longrightarrow \quad S_M(T)=\frac{E_M(T)-F_M(T)}{T}$,

Thermodynamic Quantities of AIT: **Rigorous Definitions**

Redefine the transient definitions rigorously as follows.

Definition Let $q_1, q_2, q_3, \dots\dots$ be an arbitrary enumeration of Dom M. □

Note that the results of this talk are independent of the choice of $\{q_i\}$.

Definition [Thermodynamic Quantities of AIT] Let $T > 0$.

(i) partition function $Z_M(T) := \lim_{m\to\infty} Z_m(T)$, where $Z_m(T) = \sum_{i=1}^{m} 2^{-\frac{|q_i|}{T}}$.

(ii) free energy $F_M(T) := \lim_{m\to\infty} F_m(T)$, where $F_m(T) = -T \log_2 Z_m(T)$.

(ii) energy $E_M(T) := \lim_{m\to\infty} E_m(T)$, where $E_m(T) = \frac{1}{Z_m(T)} \sum_{i=1}^{m} |q_i|\, 2^{-\frac{|q_i|}{T}}$.

(iii) entropy $S_M(T) := \lim_{m\to\infty} S_m(T)$, where $S_m(T) = \frac{E_m(T) - F_m(T)}{T}$. □

Remark These are variants of Chaitin's Ω in the case where M is optimal.

Partial Randomness

Partial Randomness of Real Number

Let $\alpha \in \mathbb{R}$. $\alpha\restriction_n$ denotes the first n bits of the base-two expansion of $\alpha - \lfloor\alpha\rfloor$.

The fractional part of α.

Definition [weak Chaitin T-randomness] Let $T \in [0,1]$.

We say $\alpha \in \mathbb{R}$ is <u>*weakly Chaitin T-random*</u> if $Tn \le H(\alpha\restriction_n) + O(1)$ for $\forall\, n$.

In the case of $T = 1$, the weak Chaitin T-randomness of a real α is equivalent to the Martin-Löf randomness (Schnorr).

Definition [T-compressibility] Let $T \in [0,1]$.

We say $\alpha \in \mathbb{R}$ is <u>*T-compressible*</u> if $H(\alpha\restriction_n) \le Tn + o(n)$,

which is equivalent to $\limsup_{n\to\infty} \frac{H(\alpha\restriction_n)}{n} \le T$.

Remark If $\alpha \in \mathbb{R}$ is weakly Chaitin T-random and T-compressible, then

$$\lim_{n\to\infty} \frac{H(\alpha\restriction_n)}{n} = T.$$

The <u>*compression rate*</u> of α by program-size complexity is equal to T.

⟨The converse does not necessarily hold.⟩

Thermodynamic Quantities of AIT

Thermodynamic Quantities of AIT – **Properties**

Theorem [properties of $Z(T)$ and $F(T)$] Let T be a real, and let V be an optimal prefix-free machine.
(i) If $0 < T \leq 1$ and T is computable, then each of $Z_V(T)$ and $F_V(T)$ converges to a real which is weakly Chaitin T-random and T-compressible.
(ii) If $1 < T$, then $Z_V(T)$ diverges to ∞, and $F_V(T)$ diverges to $-\infty$. $\square$

Definition We say $\alpha \in \mathbb{R}$ is *Chaitin T-random* if $\lim_{n\to\infty} H(\alpha\restriction_n) - Tn = \infty$.

Theorem [properties of $E(T)$ and $S(T)$] Let T be a real, and let V be an optimal prefix-free machine.
(i) If $0 < T < 1$ and T is computable, then each of $E_V(T)$ and $S_V(T)$ converges to a real which is Chaitin T-random and T-compressible.
(ii) If $1 \leq T$, then $E_V(T)$ and $S_V(T)$ diverge to ∞. $\square$

Implication of the part (i) of the above theorems: The partial randomness (and therefore compression rate) of all the thermodynamic quantities at temperature T equals to the temperature T for every computable $T \in (0, 1)$.

Fixed Point Theorem on Partial Randomness

Theorem [fixed point theorem on partial randomness by $Z(T)$ and $F(T)$]

Let V be an optimal prefix-free machine. For every $T \in (0,1)$, if one of $Z_V(T)$ and $F_V(T)$ is a computable real, then

(i) T is weakly Chaitin T-random and T-compressible, and therefore

$\Longrightarrow$ The partial randomness of T equals to T itself.

(ii) $\lim_{n\to\infty} \dfrac{H(T\restriction_n)}{n} = T$.

$\Longrightarrow$ The compression rate of T equals to T itself.

Intuitive Meaning; Metaphor

Consider a file of infinite size whose content is

"The compression rate of this file is $0.100111001\cdots\cdots$"

When this file is compressed, the compression rate of this file actually equals to $0.100111001\cdots\cdots$, as the content of this file says.

This situation forms a fixed point and is self-referential !

Fixed Point Theorems by $E(T)$ and $S(T)$

In the fixed point theorem, the role of $Z_V(T)$ and $F_V(T)$ can be replaced by the thermodynamic quantities $E_V(T)$ and $S_V(T)$ as follows.

Theorem [fixed point theorem by $E(T)$ and $S(T)$] Let V be an optimal prefix-free machine. For every $T \in (0,1)$, if one of $E_V(T)$ and $S_V(T)$ is computable, then

(i) T is Chaitin T-random and T-compressible, and therefore

(ii) $\lim_{n\to\infty} H(T\restriction_n)/n = T$. □

This fixed point theorem has the same (a stronger) form as by $Z(T)$ and $F(T)$.

Investigation of Fixed Points

Investigation of Fixed Points

Let V be an optimal prefix-free machine. We define a set $\mathcal{Z}(V)$ as

$$\mathcal{Z}(V) := \{\, T \in (0,1) \mid Z_V(T) \text{ is computable} \,\}.$$

In the same manner, we define the sets $\mathcal{F}(V)$, $\mathcal{E}(V)$, and $\mathcal{S}(V)$ based on the computability of $F_V(T)$, $E_V(T)$, and $S_V(T)$, respectively.

In this talk, we investigate the disjointness properties of the sufficient conditions of the fixed point theorems by showing the following theorem.

Theorem [simultaneous disjointness] There exists a recursive enumeration $V_1, V_2, V_3, \dots$ of optimal prefix-free machines such that

$$\mathcal{Z}(V_k) \cap \mathcal{Z}(V_l) = \mathcal{F}(V_k) \cap \mathcal{F}(V_l) = \mathcal{E}(V_k) \cap \mathcal{E}(V_l) = \mathcal{S}(V_k) \cap \mathcal{S}(V_l) = \emptyset$$

for all k, l with $k \neq l$. $\square$

This theorem is obtained by developing the statistical mechanical interpretation of AIT further and pursuing its formal correspondence to normal statistical mechanics.

Composition of Systems in Normal Statistical Mechanics

Recall the fundamental fact of statistical mechanics (or of thermodynamics):

Consider N quantum systems $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$.

These N quantum systems form a large one quantum system $\mathcal{Q}_{1\otimes\cdots\otimes N}$, a composite system.

Suppose that the ith quantum system $\mathcal{Q}_i$ has the free energy F_i, energy E_i, and entropy S_i. Then the free energy $F_{1\otimes\cdots\otimes N}$, energy $E_{1\otimes\cdots\otimes N}$, and entropy $S_{1\otimes\cdots\otimes N}$ of the composite system $\mathcal{Q}_{1\otimes\cdots\otimes N}$ are given as

$$
\begin{aligned}
F_{1\otimes\cdots\otimes N} &= F_1 + \cdots + F_N, \\
E_{1\otimes\cdots\otimes N} &= E_1 + \cdots + E_N, \\
S_{1\otimes\cdots\otimes N} &= S_1 + \cdots + S_N.
\end{aligned}
$$

Thus, the notion of free energy, energy, and entropy have additivity, and therefore are called *extensive parameters*.

Composition of Systems in AIT

In the statistical mechanical interpretation of AIT, each prefix-free machine corresponds to a quantum system in quantum mechanics.

Inspired by the notion of composition of quantum systems in quantum mechanics, we introduce the notion of composition of prefix-free machines into AIT as follows.

Definition [composition of prefix-free machines] Let $M_1, M_2, \dots, M_N$ be prefix-free machines.
The *composition* of M_1, M_2, ..., and M_N, denoted by $M_1 \oslash M_2 \oslash \cdots \oslash M_N$, is defined as the prefix-free machine D such that

(i) $\mathrm{Dom}\, D = \{p_1 p_2 \dots p_N \mid p_1 \in \mathrm{Dom}\, M_1 \ \& \ \cdots \ \& \ p_N \in \mathrm{Dom}\, M_N\}$,

(ii) $D(p_1 p_2 \dots p_N) = M_1(p_1)$. □

Theorem A Let $M_1, M_2, \dots, M_N$ be prefix-free machines. If M_1 is optimal then the composition $M_1 \oslash M_2 \oslash \cdots \oslash M_N$ is also an optimal prefix-free machine. □

Thermodynamic Quantities in AIT for the Composition of machines

In the same manner as in normal statistical mechanics, we can prove the following theorem for the thermodynamic quantities in AIT.

Theorem B Let $M_1, M_2, \ldots, M_N$ be prefix-free machines. Then the following hold:

$$
\begin{aligned}
F_{M_1 \oslash \cdots \oslash M_N}(T) &= F_{M_1}(T) + \cdots + F_{M_N}(T), \\
E_{M_1 \oslash \cdots \oslash M_N}(T) &= E_{M_1}(T) + \cdots + E_{M_N}(T), \\
S_{M_1 \oslash \cdots \oslash M_N}(T) &= S_{M_1}(T) + \cdots + S_{M_N}(T).
\end{aligned}
$$

$\square$

These equations recover the fact that free energy, energy, and entropy are extensive parameters in normal statistical mechanics.

Computable Measure Machines

Definition [computable measure machine, Downey & Griffiths 2004]

A prefix-free machine M is called a *computable measure machine* if

$$Z_M(1) := \sum_{p \in \mathrm{Dom}\, M} 2^{-|p|}$$

is a computable real. $\square$

Theorem C

Let M be a computable measure machine. Then, for every $T \in (0,1)$, the following conditions are equivalent:

(i) T is computable.

(ii) At least, one of $Z_M(T)$, $F_M(T)$, $E_M(T)$, and $S_M(T)$ is computable.

(iii) All of $Z_M(T)$, $F_M(T)$, $E_M(T)$, and $S_M(T)$ are computable. $\square$

The implication (ii) $\Rightarrow$ (i) plays a crusial role in the proof of the theorem.

Simple Examples Which Have a Physical Analogue

Example [computable measure machines]

(i) Two level system (i.e. Qubit):

Let B be a particular prefix-free machine for which $\mathrm{Dom}\, B = \{1, 01\}$. Then,

$$
\begin{aligned}
Z_B(T) &= 2^{-1/T} + 2^{-2/T}, \\
F_B(T) &= -T \log_2 Z_B(T), \\
E_B(T) &= \frac{1}{Z_B(T)} \left(2^{-1/T} + 2 \cdot 2^{-2/T}\right), \\
S_B(T) &= (E_B(T) - F_B(T))/T.
\end{aligned}
$$

(ii) One dimensional harmonic oscillator:

Let O be a particular prefix-free machine for which $\mathrm{Dom}\, O = \{0^l 1 \mid l \in \mathbb{N}\}$. Then,

$$
\begin{aligned}
Z_O(T) &= \frac{1}{2^{1/T} - 1}, \\
F_O(T) &= T \log_2 \left(2^{1/T} - 1\right), \\
E_O(T) &= \frac{2^{1/T}}{2^{1/T} - 1} \qquad \text{“Planck Radiation Formula”}, \\
S_O(T) &= (E_O(T) - F_O(T))/T.
\end{aligned}
$$

Proof of the Result

Theorem [simultaneous disjointness, posted again] There exists a recursive enumeration $V_1, V_2, V_3, \dots$ of optimal prefix-free machines such that

$$\mathcal{Z}(V_k) \cap \mathcal{Z}(V_l) = \mathcal{F}(V_k) \cap \mathcal{F}(V_l) = \mathcal{E}(V_k) \cap \mathcal{E}(V_l) = \mathcal{S}(V_k) \cap \mathcal{S}(V_l) = \emptyset$$

for all k, l with $k \neq l$.

Proof) Choose a particular optimal prefix-free machine U and a particular computable measure machine M. Then, for each $k \in \mathbb{N}^+$, define the prefix-free machine V_k by

$$V_k := U \oslash \underbrace{M \oslash \cdots \oslash M}_{k}.$$

Then, by Theorem A, V_k is optimal for every k. Using Theorem B,

$$E_{V_k}(T) = E_U(T) + kE_M(T).$$

Let k and l be arbitrary two positive integers with $k > l$. Then,

$$E_{V_k}(T) = E_{V_l}(T) + (k-l)E_M(T).$$

If both $E_{V_k}(T)$ and $E_{V_l}(T)$ are computable, then $E_M(T)$ is computable, and therefore T is also computable by Theorem C. However, ... a contradiction. □

Thank you !