A statistical mechanical interpretation of algorithmic information theory

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Abstract. We develop a statistical mechanical interpretation of algorithmic information theory by introducing the notion of thermodynamic quantities, such as free energy, energy, statistical mechanical entropy, and specific heat, into algorithmic information theory. We investigate the properties of these quantities by means of program-size complexity from the point of view of algorithmic randomness. It is then discovered that, in the interpretation, the temperature plays a role as the compression rate of the values of all these thermodynamic quantities, which include the temperature itself. Reflecting this self-referential nature of the compression rate of the temperature, we obtain fixed point theorems on compression rate.

Key words: algorithmic information theory, algorithmic randomness, Chaitin's Ω, compression rate, fixed point theorem, statistical mechanics, temperature

1 Introduction

Algorithmic information theory is a framework to apply information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of algorithmic information theory is the *program-size complexity* (or *Kolmogorov complexity*) $H(s)$ of a finite binary string s, which is defined as the length of the shortest binary program for the universal self-delimiting Turing machine U to output s. By the definition, $H(s)$ can be thought of as the information content of the individual finite binary string s. In fact, algorithmic information theory has precisely the formal properties of classical information theory (see Chaitin [3]). The concept of program-size complexity plays a crucial role in characterizing the randomness of a finite or infinite binary string. In [3] Chaitin introduced the halting probability Ω as an example of random infinite binary string. His Ω is defined as the probability that the universal self-delimiting Turing machine U halts, and plays a central role in the metamathematical development of algorithmic information theory. The first n bits of the base-two expansion of Ω solves the halting problem for a program of size not greater than n. By this property, the base-two expansion of Ω is shown to be a random infinite binary string.

In [7, 8] we generalized Chaitin's halting probability Ω to Ω^D by

$$\Omega^D = \sum_{p \in \mathrm{dom}\, U} 2^{-\frac{|p|}{D}}, \tag{1}$$

so that the degree of randomness of Ω^D can be controlled by a real number D with $0 < D \leq 1$. Here, $\operatorname{dom} U$ denotes the set of all programs p for U. As D becomes larger, the degree of randomness of Ω^D increases. When $D = 1$, Ω^D becomes a random real number, i.e., $\Omega^1 = \Omega$. The properties of Ω^D and its relations to self-similar sets were studied in Tadaki [7, 8].

Recently, Calude and Stay [2] pointed out a formal correspondence between Ω^D and a partition function in statistical mechanics. In statistical mechanics, the partition function $Z(T)$ at temperature T is defined by

$$Z(T) = \sum_{x \in X} e^{-\frac{E_x}{kT}},$$

where X is a complete set of energy eigenstates of a statistical mechanical system and E_x is the energy of an energy eigenstate x. The constant k is called the Boltzmann Constant. The partition function $Z(T)$ is of particular importance in equilibrium statistical mechanics. This is because all the thermodynamic quantities of the system can be expressed by using the partition function $Z(T)$, and the knowledge of $Z(T)$ is sufficient to understand all the macroscopic properties of the system. Calude and Stay [2] pointed out, in essence, that the partition function $Z(T)$ has the same form as Ω^D by performing the following replacements in $Z(T)$:

Replacements 1

(i) Replace the complete set X of energy eigenstates x by the set $\operatorname{dom} U$ of all programs p for U.
(ii) Replace the energy E_x of an energy eigenstate x by the length $|p|$ of a program p.
(iii) Set the Boltzmann Constant k to $1/\ln 2$, where the $\ln$ denotes the natural logarithm. □

In this paper, inspired by their suggestion above, we develop a statistical mechanical interpretation of algorithmic information theory, where Ω^D appears as a partition function.

Generally speaking, in order to give a statistical mechanical interpretation to a framework which looks unrelated to statistical mechanics at first glance, it is important to identify a microcanonical ensemble in the framework. Once we can do so, we can easily develop an equilibrium statistical mechanics on the framework according to the theoretical development of normal equilibrium statistical mechanics. Here, the microcanonical ensemble is a certain sort of uniform probability distribution. In fact, in the work [9] we developed a statistical mechanical interpretation of the noiseless source coding scheme in information theory by identifying a microcanonical ensemble in the scheme. Then, in [9] the notions in statistical mechanics such as statistical mechanical entropy, temperature, and thermal equilibrium are translated into the context of noiseless source coding.

Thus, in order to develop a statistical mechanical interpretation of algorithmic information theory, it is appropriate to identify a microcanonical ensemble in the framework of the theory. Note, however, that algorithmic information theory is not a physical theory but a purely mathematical theory. Therefore, in order to obtain significant results for the development of algorithmic information theory itself, we have to develop a statistical mechanical interpretation of algorithmic information theory in a mathematically rigorous manner, unlike in normal statistical mechanics in physics where arguments are not necessarily mathematically rigorous. A fully rigorous mathematical treatment of statistical mechanics is already developed (see Ruelle [6]). At present, however, it would not as yet seem to be an easy task to merge algorithmic information theory with this mathematical treatment in a satisfactory manner.

On the other hand, if we do not stick to the mathematical strictness of an argument and make an argument on the same level of mathematical strictness as statistical mechanics in physics, we can develop a statistical mechanical interpretation of algorithmic information theory while realizing a perfect correspondence to normal statistical mechanics. In the physical argument, we can identify a microcanonical ensemble in algorithmic information theory in a similar manner to [9], based on the probability measure which gives Chaitin's Ω the meaning of the halting probability actually.[1] In consequence, for example, the statistical mechanical meaning of Ω^D is clarified.

In this paper, we develop a statistical mechanical interpretation of algorithmic information theory in a different way from the physical argument mentioned above.[2] We introduce the notion of thermodynamic quantities into algorithmic information theory based on Replacements 1 above.

After the preliminary section on the mathematical notion needed in this paper, in Section 3 we introduce the notion of the thermodynamic quantities at any given fixed temperature T, such as partition function, free energy, energy, statistical mechanical entropy, and specific heat, into algorithmic information theory by performing Replacements 1 for the corresponding thermodynamic quantities in statistical mechanics. These thermodynamic quantities in algorithmic information theory are real numbers which depend only on the temperature T. We prove that if the temperature T is a computable real number with $0 < T < 1$ then, for each of these thermodynamic quantities, the compression rate by the program-size complexity H is equal to T. Thus, the temperature T plays a role as the compression rate of the thermodynamic quantities in this statistical mechanical interpretation of algorithmic information theory.

Among all thermodynamic quantities in thermodynamics, one of the most typical thermodynamic quantities is temperature itself. Thus, based on the results of Section 3, the following question naturally arises: Can the compression rate of the temperature T be equal to the temperature itself in the statistical mechanical interpretation of algorithmic information theory ? This question is rather self-referential. However, in Section 4 we answer it affirmatively by prov-

[1] Due to the 10-page limit, we omit the detail of the physical argument in this paper. It will be included in a full version of this paper, and is also available in Section 6 of an extended and electronic version of this paper at URL: `http://arxiv.org/abs/0801.4194v1`

[2] We make an argument in a fully mathematically rigorous manner in this paper.

ing Theorem 9. One consequence of Theorem 9 has the following form: For every $T \in (0,1)$, if $\Omega^T = \sum_{p \in \mathrm{dom}\, U} 2^{-\frac{|p|}{T}}$ is a computable real number, then

$$\lim_{n\to\infty} \frac{H(T_n)}{n} = T,$$

where T_n is the first n bits of the base-two expansion of T. This is just a fixed point theorem on compression rate, which reflects the self-referential nature of the question.

The works [7, 8] on Ω^D might be regarded as an elaboration of the technique used by Chaitin [3] to prove that Ω is random. The results of this paper may be regarded as further elaborations of the technique. Due to the 10-page limit, we omit most proofs. A full paper describing the details of the proofs is in preparation.[3]

2 Preliminaries

We start with some notation about numbers and strings which will be used in this paper. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers. $\{0,1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$ is the set of finite binary strings, where λ denotes the *empty string*. For any $s \in \{0,1\}^*$, $|s|$ is the *length* of s. A subset S of $\{0,1\}^*$ is called a *prefix-free set* if no string in S is a prefix of another string in S. $\{0,1\}^\infty$ is the set of infinite binary strings, where an infinite binary string is infinite to the right but finite to the left. For any $\alpha \in \{0,1\}^\infty$ and any $n \in \mathbb{N}^+$, α_n is the prefix of α of length n. For any partial function f, the domain of definition of f is denoted by $\mathrm{dom}\, f$. We write "r.e." instead of "recursively enumerable."

Normally, $o(n)$ denotes any function $f\colon \mathbb{N}^+ \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any function $g\colon \mathbb{N}^+ \to \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \le C$ for all $n \in \mathbb{N}^+$.

Let T be an arbitrary real number. $T \bmod 1$ denotes $T - \lfloor T \rfloor$, where $\lfloor T \rfloor$ is the greatest integer less than or equal to T. Hence, $T \bmod 1 \in [0, 1)$. We identify a real number T with the infinite binary string α such that $0.\alpha$ is the base-two expansion of $T \bmod 1$ with infinitely many zeros. Thus, T_n denotes the first n bits of the base-two expansion of $T \bmod 1$ with infinitely many zeros.

We say that a real number T is *computable* if there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $|T - f(n)| < 1/n$ for all $n \in \mathbb{N}^+$. We say that T is *right-computable* if there exists a total recursive function $g\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $T \le g(n)$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} g(n) = T$. We say that T is *left-computable* if $-T$ is right-computable. It is then easy to see that, for any $T \in \mathbb{R}$, T is computable if and only if T is both right-computable and left-computable. See e.g. Weihrauch [12] for the detail of the treatment of the computability of real numbers and real functions on a discrete set.

[3] The details of the proofs are also available in an extended and electronic version of this paper at URL: `http://arxiv.org/abs/0801.4194v1`

2.1 Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [3, 4]. A *computer* is a partial recursive function $C\colon \{0,1\}^* \to \{0,1\}^*$ such that $\operatorname{dom} C$ is a prefix-free set. For each computer C and each $s \in \{0,1\}^*$, $H_C(s)$ is defined by $H_C(s) = \min\{\,|p| \mid p \in \{0,1\}^* \,\&\, C(p) = s\,\}$. A computer U is said to be *optimal* if for each computer C there exists a constant $\operatorname{sim}(C)$ with the following property; if $C(p)$ is defined, then there is a p' for which $U(p') = C(p)$ and $|p'| \le |p| + \operatorname{sim}(C)$. It is easy to see that there exists an optimal computer. Note that the class of optimal computers equals to the class of functions which are computed by *universal self-delimiting Turing machines* (see Chaitin [3] for the detail). We choose a particular optimal computer U as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the *program-size complexity* of s or the *Kolmogorov complexity* of s.

Chaitin's halting probability Ω is defined by

$$\Omega = \sum_{p \in \operatorname{dom} U} 2^{-|p|}.$$

For any $\alpha \in \{0,1\}^\infty$, we say that α is *weakly Chaitin random* if there exists $c \in \mathbb{N}$ such that $n - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$ [3, 4]. Then Chaitin [3] showed that Ω is weakly Chaitin random. For any $\alpha \in \{0,1\}^\infty$, we say that α is *Chaitin random* if $\lim_{n\to\infty} H(\alpha_n) - n = \infty$ [3, 4]. It is then shown that, for any $\alpha \in \{0,1\}^\infty$, α is weakly Chaitin random if and only if α is Chaitin random (see Chaitin [4] for the proof and historical detail). Thus Ω is Chaitin random.

In the works [7, 8], we generalized the notion of the randomness of an infinite binary string so that the degree of the randomness can be characterized by a real number D with $0 < D \le 1$ as follows.

Definition 1 (weak Chaitin D-randomness and D-compressibility). *Let $D \in \mathbb{R}$ with $D \ge 0$, and let $\alpha \in \{0,1\}^\infty$. We say that α is weakly Chaitin D-random if there exists $c \in \mathbb{N}$ such that $Dn - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$. We say that α is D-compressible if $H(\alpha_n) \le Dn + o(n)$, which is equivalent to $\overline{\lim}_{n\to\infty} H(\alpha_n)/n \le D$.* □

In the case of $D = 1$, the weak Chaitin D-randomness results in the weak Chaitin randomness. For any $D \in [0,1]$ and any $\alpha \in \{0,1\}^\infty$, if α is weakly Chaitin D-random and D-compressible, then

$$\lim_{n\to\infty} \frac{H(\alpha_n)}{n} = D. \tag{2}$$

Hereafter the left-hand side of (2) is referred to as the *compression rate* of an infinite binary string α in general. Note, however, that (2) does not necessarily imply that α is weakly Chaitin D-random.

In the works [7, 8], we generalized Chaitin's halting probability Ω to Ω^D by (1) for any real number $D > 0$. Thus, $\Omega = \Omega^1$. If $0 < D \le 1$, then Ω^D converges and $0 < \Omega^D < 1$, since $\Omega^D \le \Omega < 1$.

Theorem 2 (Tadaki [7, 8]). *Let $D \in \mathbb{R}$.*

(i) If $0 < D \leq 1$ and D is computable, then Ω^D is weakly Chaitin D-random and D-compressible.

(ii) If $1 < D$, then Ω^D diverges to ∞. □

Definition 2 (Chaitin D-randomness, Tadaki [7, 8]). *Let $D \in \mathbb{R}$ with $D \geq 0$, and let $\alpha \in \{0,1\}^\infty$. We say that α is Chaitin D-random if $\lim_{n\to\infty} H(\alpha_n) - Dn = \infty$.* □

In the case of $D = 1$, the Chaitin D-randomness results in the Chaitin randomness. Obviously, for any $D \in [0, 1]$ and any $\alpha \in \{0,1\}^\infty$, if α is Chaitin D-random, then α is weakly Chaitin D-random. However, in 2005 Reimann and Stephan [5] showed that, in the case of $D < 1$, the converse does not necessarily hold. This contrasts with the equivalence between the weakly Chaitin randomness and the Chaitin randomness, each of which corresponds to the case of $D = 1$.

For each real numbers $Q > 0$ and $D > 0$, we define $W(Q, D)$ by

$$W(Q, D) = \sum_{p \in \operatorname{dom} U} |p|^Q \, 2^{-\frac{|p|}{D}}.$$

As the first result of this paper, we can show the following theorem.

Theorem 3. *Let Q and D be positive real numbers.*

(i) If Q and D are computable and $0 < D < 1$, then $W(Q, D)$ converges to a left-computable real number which is Chaitin D-random and D-compressible.

(ii) If $1 \leq D$, then $W(Q, D)$ diverges to ∞. □

Thus, we see that the weak Chaitin D-randomness in Theorem 2 is replaced by the Chaitin D-randomness in Theorem 3 in exchange for the divergence at $D = 1$.

3 Temperature as a compression rate

In this section we introduce the notion of thermodynamic quantities such as partition function, free energy, energy, statistical mechanical entropy, and specific heat, into algorithmic information theory by performing Replacements 1 for the corresponding thermodynamic quantities in statistical mechanics.[4] We investigate their convergence and the degree of randomness. For that purpose, we first choose a particular enumeration $q_1, q_2, q_3, \ldots$ of the countably infinite set $\operatorname{dom} U$ as the standard one for use throughout this section.[5]

[4] For the thermodynamic quantities in statistical mechanics, see Chapter 16 of [1] and Chapter 2 of [11]. To be precise, the partition function is not a thermodynamic quantity but a statistical mechanical quantity.

[5] The enumeration $\{q_i\}$ is quite arbitrary and therefore we do not, ever, require $\{q_i\}$ to be a recursive enumeration of $\operatorname{dom} U$.

In statistical mechanics, the partition function $Z_{\mathrm{sm}}(T)$ at temperature T is given by

$$Z_{\mathrm{sm}}(T) = \sum_{x \in X} e^{-\frac{E_x}{kT}}, \tag{3}$$

Motivated by the formula (3) and taking into account Replacements 1, we introduce the notion of partition function into algorithmic information theory as follows.

Definition 3 (partition function). *For each $n \in \mathbb{N}^+$ and each real number $T > 0$, we define $Z_n(T)$ by*

$$Z_n(T) = \sum_{i=1}^{n} 2^{-\frac{|q_i|}{T}}.$$

Then, the partition function $Z(T)$ is defined by $Z(T) = \lim_{n\to\infty} Z_n(T)$, for each $T > 0$. □

Since $Z(T) = \Omega^T$, we restate Theorem 2 as in the following form.

Theorem 4 (Tadaki [7, 8]). *Let $T \in \mathbb{R}$.*

(i) If $0 < T \leq 1$ and T is computable, then $Z(T)$ converges to a left-computable real number which is weakly Chaitin T-random and T-compressible.
(ii) If $1 < T$, then $Z(T)$ diverges to ∞. □

In statistical mechanics, the free energy $F_{\mathrm{sm}}(T)$ at temperature T is given by

$$F_{\mathrm{sm}}(T) = -kT \ln Z_{\mathrm{sm}}(T), \tag{4}$$

where $Z_{\mathrm{sm}}(T)$ is given by (3). Motivated by the formula (4) and taking into account Replacements 1, we introduce the notion of free energy into algorithmic information theory as follows.

Definition 4 (free energy). *For each $n \in \mathbb{N}^+$ and each real number $T > 0$, we define $F_n(T)$ by $F_n(T) = -T \log_2 Z_n(T)$. Then, for each $T > 0$, the free energy $F(T)$ is defined by $F(T) = \lim_{n\to\infty} F_n(T)$.* □

Theorem 5. *Let $T \in \mathbb{R}$.*

(i) If $0 < T \leq 1$ and T is computable, then $F(T)$ converges to a right-computable real number which is weakly Chaitin T-random and T-compressible.
(ii) If $1 < T$, then $F(T)$ diverges to $-\infty$. □

In statistical mechanics, the energy $E_{\mathrm{sm}}(T)$ at temperature T is given by

$$E_{\mathrm{sm}}(T) = \frac{1}{Z_{\mathrm{sm}}(T)} \sum_{x \in X} E_x e^{-\frac{E_x}{kT}}, \tag{5}$$

where $Z_{\mathrm{sm}}(T)$ is given by (3). Motivated by the formula (5) and taking into account Replacements 1, we introduce the notion of energy into algorithmic information theory as follows.

Definition 5 (energy). *For each $n \in \mathbb{N}^+$ and each real number $T > 0$, we define $E_n(T)$ by*

$$E_n(T) = \frac{1}{Z_n(T)} \sum_{i=1}^{n} |q_i|\, 2^{-\frac{|q_i|}{T}}.$$

Then, for each $T > 0$, the energy $E(T)$ is defined by $E(T) = \lim_{n\to\infty} E_n(T)$. □

Theorem 6. *Let $T \in \mathbb{R}$.*

(i) If $0 < T < 1$ and T is computable, then $E(T)$ converges to a left-computable real number which is Chaitin T-random and T-compressible.

(ii) If $1 \le T$, then $E(T)$ diverges to ∞. □

In statistical mechanics, the entropy $S_{\rm sm}(T)$ at temperature T is given by

$$S_{\rm sm}(T) = \frac{1}{T} E_{\rm sm}(T) + k \ln Z_{\rm sm}(T), \tag{6}$$

where $Z_{\rm sm}(T)$ and $E_{\rm sm}(T)$ are given by (3) and (5), respectively. Motivated by the formula (6) and taking into account Replacements 1, we introduce the notion of statistical mechanical entropy into algorithmic information theory as follows.

Definition 6 (statistical mechanical entropy). *For each $n \in \mathbb{N}^+$ and each real number $T > 0$, we define $S_n(T)$ by $S_n(T) = \frac{1}{T} E_n(T) + \log_2 Z_n(T)$. Then, for each $T > 0$, the statistical mechanical entropy $S(T)$ is defined by $S(T) = \lim_{n\to\infty} S_n(T)$.* □

Theorem 7. *Let $T \in \mathbb{R}$.*

(i) If $0 < T < 1$ and T is computable, then $S(T)$ converges to a left-computable real number which is Chaitin T-random and T-compressible.

(ii) If $1 \le T$, then $S(T)$ diverges to ∞. □

Finally, in statistical mechanics, the specific heat $C_{\rm sm}(T)$ at temperature T is given by

$$C_{\rm sm}(T) = \frac{d}{dT} E_{\rm sm}(T), \tag{7}$$

where $E_{\rm sm}(T)$ is given by (5). Motivated by this formula (7), we introduce the notion of specific heat into algorithmic information theory as follows.

Definition 7 (specific heat). *For each $n \in \mathbb{N}^+$ and each real number $T > 0$, we define $C_n(T)$ by $C_n(T) = E_n'(T)$, where $E_n'(T)$ is the derived function of $E_n(T)$. Then, for each $T > 0$, the specific heat $C(T)$ is defined by $C(T) = \lim_{n\to\infty} C_n(T)$.* □

Theorem 8. *Let $T \in \mathbb{R}$.*

(i) If $0 < T < 1$ and T is computable, then $C(T)$ converges to a left-computable real number which is Chaitin T-random and T-compressible, and moreover $C(T) = E'(T)$ where $E'(T)$ is the derived function of $E(T)$.

(ii) If $T = 1$, then $C(T)$ diverges to ∞. □

Thus, the theorems in this section show that the temperature T plays a role as the compression rate for all the thermodynamic quantities introduced into algorithmic information theory in this section.

These theorems also show that the values of the thermodynamic quantities: partition function, free energy, energy, and statistical mechanical entropy diverge in the case of $T > 1$. This phenomenon might be regarded as some sort of phase transition in statistical mechanics.[6]

4 Fixed point theorems on compression rate

In this section, we show the following theorem and its variant.

Theorem 9 (fixed point theorem on compression rate). *For every $T \in (0,1)$, if $Z(T)$ is a computable real number, then the following hold:*

(i) *T is right-computable and not left-computable.*
(ii) *T is weakly Chaitin T-random and T-compressible.*
(iii) $\lim_{n\to\infty} H(T_n)/n = T$. □

Theorem 9 follows immediately from the following three theorems.

Theorem 10. *For every $T \in (0,1)$, if $Z(T)$ is a right-computable real number, then T is weakly Chaitin T-random.* □

Theorem 11. *For every $T \in (0,1)$, if $Z(T)$ is a right-computable real number, then T is also a right-computable real number.* □

Theorem 12. *For every $T \in (0,1)$, if $Z(T)$ is a left-computable real number and T is a right-computable real number, then T is T-compressible.* □

Theorem 9 is just a fixed point theorem on compression rate, where the computability of the value $Z(T)$ gives a sufficient condition for a real number $T \in (0,1)$ to be a fixed point on compression rate. Note that $Z(T)$ is a strictly increasing continuous function on $(0,1)$. In fact, Tadaki [7, 8] showed that $Z(T)$ is a function of class C^∞ on $(0,1)$. Thus, since the set of all computable real numbers is dense in $\mathbb{R}$, we have the following for this sufficient condition.

Theorem 13. *The set $\{T \in (0,1) \mid Z(T)$ is computable$\}$ is dense in $[0,1]$.* □

We thus have the following corollary of Theorem 9.

Corollary 1. *The set $\{T \in (0,1) \mid \lim_{n\to\infty} H(T_n)/n = T\}$ is dense in $[0,1]$.* □

From the point of view of the statistical mechanical interpretation introduced in the previous section, Theorem 9 shows that the compression rate of temperature is equal to the temperature itself. Thus, Theorem 9 further confirms the role of temperature as the compression rate, which is observed in the previous section.

In a similar manner to the proof of Theorem 9, we can prove another version of a fixed point theorem on compression rate as follows. Here, the weak Chaitin T-randomness is replaced by the Chaitin T-randomness.

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[6] It is still open whether $C(T)$ diverges or not in the case of $T > 1$.

Theorem 14 (fixed point theorem on compression rate II). *Let Q be a computable real number with $Q > 0$. For every $T \in (0,1)$, if $W(Q,T)$ is a computable real number, then the following hold:*

(i) T is right-computable and not left-computable.
(ii) T is Chaitin T-random and T-compressible. □

For the sufficient condition of Theorem 14, in a similar manner to the case of Theorem 9, we can show that, for every $Q > 0$, the set $\{T \in (0,1) \mid W(Q,T)$ is computable$\}$ is dense in $[0,1]$.

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