A statistical mechanical interpretation of algorithmic information theory

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$\boxed{\text{Abstract: What we will do in this talk}}$ U: universal Turing machine.

Algorithmic Information Theory (AIT, for short) is a theory of program-size. In this talk, we introduce the notion of thermodynamic quantities into AIT by performing the following replacements for the thermodynamic quantities of a quantum system at temperature T obeying the canonical distribution. We then investigate their randomness properties.

An energy eigenstate n	$\Longrightarrow$	A program p of U,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p\rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$.

Boltzmann factor: $2^{-\frac{|p|}{T}}$

Partition function $$Z(T)=\sum_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad Z(T)=\sum_p 2^{-\frac{|p|}{T}},$$

Free energy $$F(T)=-kT\ln Z(T) \quad \Longrightarrow \quad F(T)=-T\log_2 Z(T),$$

Energy $$E(T)=\frac{1}{Z(T)}\sum_n E_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad E(T)=\frac{1}{Z(T)}\sum_p |p|\, 2^{-\frac{|p|}{T}},$$

Entropy $$S(T)=\frac{E(T)-F(T)}{T} \quad \Longrightarrow \quad S(T)=\frac{E(T)-F(T)}{T}.$$

Temperature = Compression Rate.

Preliminaries: Prefix-free Sets

- $\{0,1\}^* := \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$
  The set of finite binary strings
  Here, λ denotes *the empty string*.

- For any $s \in \{0,1\}^*$, $|s|$ denotes the *length* of s.

For example $|010| = 3$, $|\lambda| = 0$.

- Let P be a subset of $\{0,1\}^*$. We say P is *prefix-free* if for any distinct s and $t \in P$, s is not a prefix of t.

For example $\{0, 10\}$: prefix-free.
$\{0, 01\}$: not prefix-free.

- A prefix-free set can be finite and can be infinite.

- For every prefix-free set $P \subset \{0,1\}^*$,

$$\sum_{s \in P} 2^{-|s|} \leq 1. \qquad \text{(Kraft inequality)}$$

Preliminaries: Prefix of Real Number

Definition [a prefix of a real]

Let α be a real, and let n be a positive integer.
We denote by $\alpha\restriction_n$ the first n bits of the base-two expansion of $\alpha - \lfloor\alpha\rfloor$.

The fractional part of α.

Example [circle ratio]

Consider the case of $\alpha = \pi$, *the circle ratio*.

$$\pi = 11.0010010000111111101\ldots\ldots$$

in base-two notation.
Therefore, the fractional part of π,

$$\pi - \lfloor\pi\rfloor = .0010010000111111101\ldots\ldots$$

in base-two notation.
Thus,

$$\pi\restriction_1 = 0, \quad \pi\restriction_2 = 00, \quad \pi\restriction_3 = 001, \quad \pi\restriction_4 = 0010, \quad \ldots\ldots$$

These are finite binary strings.

Preliminaries: Partial Recursive Functions – **Definition**

Definition [Partial recursive function]

We say f is a *partial recursive function* if the following two hold for f:

(i) There exists a subset V of $\{0,1\}^*$ such that $f \colon V \to \{0,1\}^*$,
where V is the domain of definition of f and denoted by $\mathrm{Dom}\, f$.

(ii) There exists an algorithm $\mathcal{A}$ such that, for each $s \in \{0,1\}^*$,
when executing $\mathcal{A}$ with the input s,

if $s \in \mathrm{Dom}\, f$ then the computation of $\mathcal{A}$ terminates and outputs $f(s)$;

if $s \notin \mathrm{Dom}\, f$ then the computation of $\mathcal{A}$ does not terminate. $\square$

Here, we can regard an algorithm, for example, as a program written by the programing language C.

Preliminaries: Partial Recursive Functions – Example

Example [a partial recursive function calculating a prefix of π]

We define the function $f_\pi\colon V_\pi \to \{0,1\}^*$ by the conditions that

(i) For every positive integer n,

$$f_\pi(\text{the base-two representation of } n) = \pi\restriction_n,$$

i.e.,

$$f_\pi(1) = \pi\restriction_1 = 0,\ f_\pi(10) = \pi\restriction_2 = 00,\ f_\pi(11) = \pi\restriction_3 = 001,\ \dots\dots$$

(ii) $\mathrm{Dom}\, f_\pi (= V_\pi)$ is the set of the base-two representations of all positive integers, i.e., $\mathrm{Dom}\, f_\pi = \{1, 10, 11, 100, 101, 110, 111, 1000, \dots\}$.

Obviously, there exists an algorithm $\mathcal{A}_\pi$ such that, given the base-two representation of a positive integer n as an input, the computation of $\mathcal{A}_\pi$ terminates and outputs $\pi\restriction_n$.

For any $s \notin \mathrm{Dom}\, f_\pi$, $\mathcal{A}_\pi$ with the input s can be made unterminated, for example, by writing the code `while(1);` on an appropriate place of the program in the case of C.

Thus, f_π is a partial recursive function. $\square$

Preliminaries: Computers

Definition [computer] A partial recursive function C is called a *computer* if $\mathrm{Dom}\, C$ is a prefix-free set. □

Example

The partial recursive function f_π is not a computer because $1,\ 10 \in \mathrm{Dom}\, f_\pi = \{1, 10, 11, 100, 101, 110, \dots\}$.

A computer C_π calculating a prefix of π is constructed as follows:

Let $1b_1b_2\dots b_{l-1}b_l$ be the base-two representation of any positive integer n, where $b_i = 0$ or 1. Then define $\overline{n} := 10b_10b_20\dots b_{l-1}0b_l1$.

We define the function C_π by the condition that, for every positive integer n, $C_\pi(\overline{n}) = \pi\restriction_n$, i.e.,

$$C_\pi(11) = \pi\restriction_1 = 0,\ C_\pi(1001) = \pi\restriction_2 = 00,\ C_\pi(1011) = \pi\restriction_3 = 001,\ \dots\dots$$

Thus, by this prescription, $\mathrm{Dom}\, C_\pi$ is made prefix-free, and therefore C_π is a computer calculating a prefix of π. □

Preliminaries: Program-size Complexity I

Definition For any computer C and any $s \in \{0,1\}^*$,

$$H_C(s) := \min\left\{ |p| \;\middle|\; p \in \{0,1\}^* \;\&\; C(p) = s \right\}.$$

Example Since $C_\pi(\overline{n}) = \pi\restriction_n$ and therefore C_π is a one-to-one function,

$$H_{C_\pi}(\pi\restriction_n) = |\overline{n}| \leq 2\log_2 n + 2.$$

Preliminaries: Program-size Complexity II

Definition [optimal computer] A computer U is called *optimal* if, for each computer C, there exists a constant $d(U,C)$ such that, for every $s \in \{0,1\}^*$,

$$H_U(s) \leq H_C(s) + d(U,C).$$

Theorem There exists an optimal computer. (a universal Turing machine)

Definition [program-size complexity]
We choose a particular optimal computer U as a standard one. Then the *program-size complexity* (or *Kolmogorov complexity*) $H(s)$ of $s \in \{0,1\}^*$ is defined by $H(s) := H_U(s)$.

Thus $H(s) \leq H_C(s) + d(U,C)$ for all computers C. Therefore, $H(s)$ can achieve the optimal compression of every $s \in \{0,1\}^*$, up to an additive constant $d(U,C)$ independent of s.
$\Longrightarrow$ $H(s)$: The amount of randomness contained in s, which cannot be captured and cannot be generated in a computational manner.

Example $H(\pi\restriction_n) \leq H_{C_\pi}(\pi\restriction_n) + O(1) \leq 2\log_2 n + O(1)$ for all n.

Preliminaries: Compression Rate of Real Number I

Definition [the compression rate of a real] Let α be a real. The limit value

$$\lim_{n\to\infty} \frac{H(\alpha\restriction_n)}{n}$$

is called the *compression rate of α*. $\square$

Numerator $H(\alpha\restriction_n)$: The size of compressed $\alpha\restriction_n$ by H.
Denominator $n = |\alpha\restriction_n|$: The original size of $\alpha\restriction_n$.

Example

A real D is called *computable* if there exists an algorithm which calculates each bit in the base-two expansion of D one by one.

- π and e are computable reals.
- Algebraic numbers and therefore rational numbers are computable reals.
- For every computable real D, $H(D\restriction_n) \le 2\log_2 n + O(1)$ for all n. Therefore, the compression rate of every computable real D equals to 0, since

$$0 \le \lim_{n\to\infty} \frac{H(D\restriction_n)}{n} \le \lim_{n\to\infty} \frac{2\log_2 n + O(1)}{n} = 0.$$

Preliminaries: Compression Rate of Real Number II

Definition [Chaitin's halting probability Ω, Chaitin 1975]

$$\Omega := \sum_{p \in \mathrm{Dom}\, U} 2^{-|p|}.$$

The first n bits of the base-two expansion of Ω (i.e., $\Omega\restriction_n$) solve the halting problem of U for inputs of length at most n.

Theorem [Chaitin 1975] Ω is incompressible, i.e., the compression rate of Ω equals to 1. $\square$

Remark Since $\mathrm{Dom}\, U$ is prefix-free, $\Omega \le 1$ by Kraft inequality and, in particular, Ω converges. This is one of the reasons why the domain of definition of a computer is required to be prefix-free.

Example Let $\Omega = 0.b_1 b_2 b_3 b_4 b_5 \dots\dots$ be the base-two expansion of Ω.

Then consider the real $\overline{\Omega} := 0.b_1 0 b_2 0 b_3 0 b_4 0 b_5 0 \dots\dots$. We can show that

$$\lim_{n\to\infty} \frac{H(\overline{\Omega}\restriction_n)}{n} = \frac{1}{2},$$

i.e., the compression rate of $\overline{\Omega}$ equals to $1/2$. $\square$

Preliminaries: Generalization of Ω

Definition [generalization of Chaitin's Ω, Tadaki 1999]

$$\Omega(D) := \sum_{p \in \mathrm{Dom}\, U} 2^{-\frac{|p|}{D}} \qquad (D > 0).$$

$\Omega(1) = \Omega$.

The first n bits of the base-two expansion of $\Omega(D)$ (i.e., $\Omega(D)\restriction_n$) solve the halting problem of U for inputs of length at most Dn.

Theorem [Tadaki 1999] Let D be a real.

(i) If $0 < D \le 1$ and D is computable, then

$$\lim_{n\to\infty} \frac{H(\Omega(D)\restriction_n)}{n} = D,$$

i.e., the compression rate of $\Omega(D)$ equals to D.

(ii) If $1 < D$, then $\Omega(D)$ diverges to ∞. $\square$

Motivation

[Calude & Stay, Information and Computation 204 (2006)] pointed out that $\Omega(D)$ is similar to a partition function in statistical mechanics.

- In statistical mechanics, the partition function Z is given as:

$$Z = \sum_{n} e^{-\frac{E_n}{kT}}.$$

Here, n denotes the quantum number of an energy eigenstate of a quantum system, E_n its energy, and T temperature.

- On the other hand, $\Omega(D)$ is given as:

$$\Omega(D) = \sum_{p \in \operatorname{Dom} U} 2^{-\frac{|p|}{D}} \qquad (D > 0).$$

Thus, Z coincides with $\Omega(D)$ by performing the following replacements:

An energy eigenstate n	$\Longrightarrow$	A program $p \in \operatorname{Dom} U$,
The energy E_n of n	$\Longrightarrow$	The length $|p|$ of p,
Temperature T	$\Longrightarrow$	Compression rate D,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$.

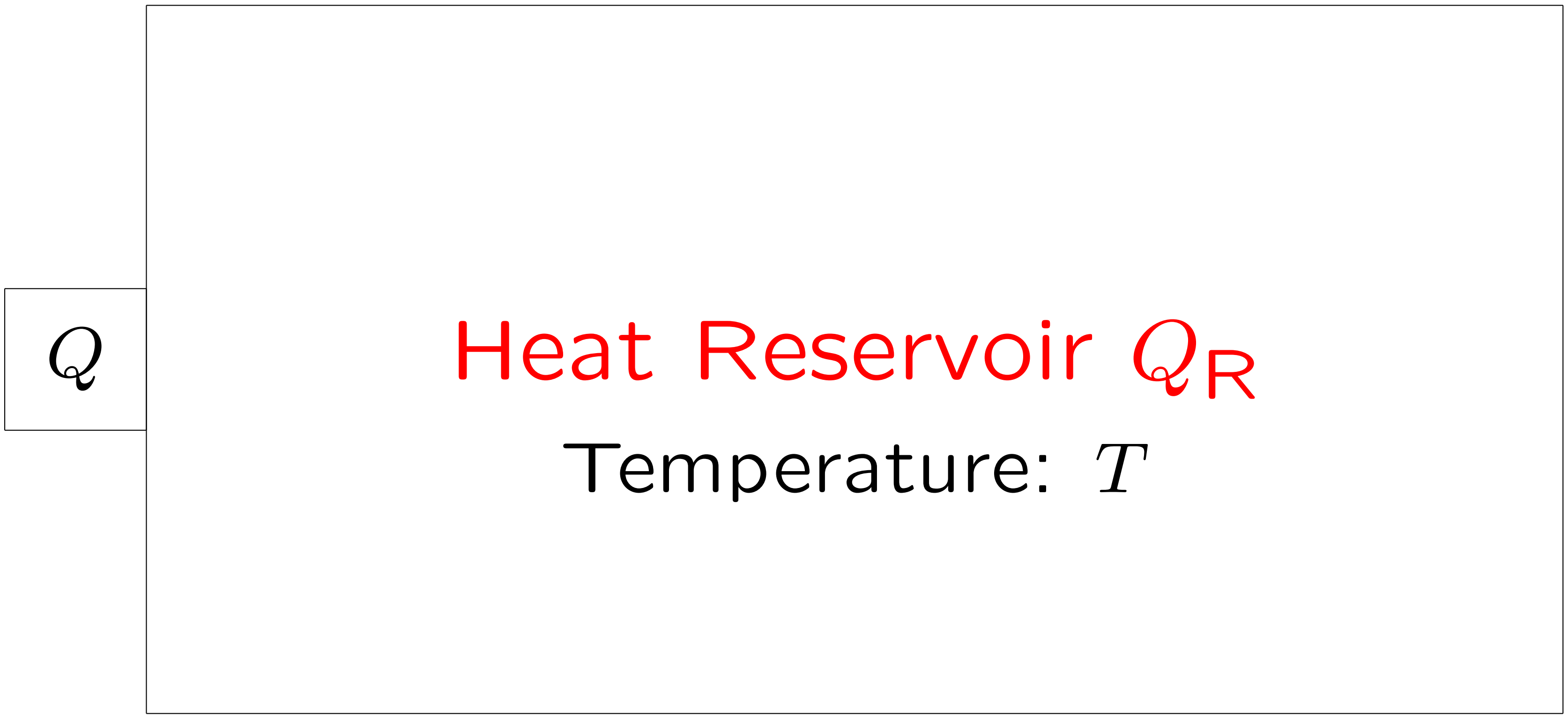
What is the partition function in statistical mechanics ?

Quick Review of Statistical Mechanics

Quick Review of Statistical Mechanics (I)

Consider a quantum system Q at constant temperature T.

Namely, consider a quantum system Q in thermal contact with a very large quantum system Q_{R}, called *heat reservoir*, whose temperature is T.



Let Q_{total} be the total quantum system consisting of Q and Q_{R}.

Quick Review of Statistical Mechanics (II)

Basic Ingredients in Statistical Mechanics

• In quantum mechanics,
any quantum system is described by a *quantum state* completely.

• In statistical mechanics,
among all quantum states, *energy eigenstates* are of particular importance.

• An energy eigenstate of a quantum system is specified by a number $n = 1, 2, 3, \ldots$, called a *quantum number.*

Thus the energy of a quantum system is assumed to take discrete values. We identify a quantum number with the corresponding energy eigenstate.

• If a quantum system is in the energy eigenstate, then the quantum system has a definite energy.

Quick Review of Statistical Mechanics (III)

Definition [(Statistical Mechanical) Entropy]

The *entropy* $S(E)$ of a quantum system with energy E is defined by

$$S(E) := k \ln \Theta(E).$$

Here, $\Theta(E)$ is the number of energy eigenstates whose energy E' satisfies that

$$E \leq E' \leq E + \delta E,$$

where δE is the indeterminacy in measurement of the energy of the quantum system. The proportional constant k is called *the Boltzmann constant*. □

Definition [Temperature]

The *temperature* T of a quantum system with energy E is defined by

$$\frac{1}{T} = \frac{\partial S}{\partial E}(E).$$

□

Note that the above definitions apply to each of the quantum systems Q, Q_{R}, and Q_{total}.

Quick Review of Statistical Mechanics (IV)

The fundamental postulate of statistical mechanics is stated as follows for the total quantum system Q_{total}:

The Principle of Equal Probability

If the energy of the quantum system Q_{total} is known to have a constant value between E and $E + \delta E$;

$$E \le (\text{The energy of } Q_{\text{total}}) \le E + \delta E,$$

then the quantum system Q_{total} is equally likely to be in any energy eigenstate whose energy E' satisfies that

$$E \le E' \le E + \delta E.$$

Here, δE is the indeterminacy in measurement of the energy of the quantum system Q_{total}. □

Quick Review of Statistical Mechanics (V)

Let us calculate the probability Prob(n) that the quantum system Q is in an energy eigenstate n with energy E_n, based on the Principle of Equal Probability for the total quantum system Q_{total} with the total energy E.

Q

Energy: E_n

Heat Reservoir Q_{R}

Energy : $E - E_n$

(by the law of the conservation of energy)

By the Principle of Equal Probability, Prob(n) is proportinal to the number $\Theta_{\text{R}}(E - E_n)$ of the energy eigenstates allowable in the heat reservoir Q_{R} with the energy $E - E_n$.

Using (i) the definition $S_{\text{R}}(E) = k \ln \Theta_{\text{R}}(E)$ of the entropy of the heat reservoir Q_{R} (ii) the definition $1/T = \frac{\partial S_{\text{R}}}{\partial E}(E)$ of the heat reservoir Q_{R}, and (iii) the fact that $E_n \ll E$, we have the following result:

Quick Review of Statistical Mechanics (VI)

As a result of the principle of equal probability, we can show the following for the quantum system Q (and not for Q_{total}):

Result of the Principle of Equal Probability

The probability Prob(n) that the quantum system Q is in an energy eigenstate n with energy E_n is given as:

$$\text{Prob}(n) = \frac{1}{Z} e^{-\frac{E_n}{kT}}.$$

Here, the normalization factor $Z := \sum_n e^{-\frac{E_n}{kT}}$ is called the *partition function* of the quantum system. The distribution Prob(n) is called *the canonical distribution*. □

The partition function Z is of particular importance in statistical mechanics, because all the thermodynamic quantities of the quantum system can be expressed by using the partition function Z, and the knowledge of Z is sufficient to understand all the macroscopic properties of the system.

Quick Review of Statistical Mechanics (VII)

Thermodynamic Quantities of the quantum system Q at temperature T

- **Energy** $E = \sum_n E_n \, \mathrm{Prob}(n) = \frac{1}{Z} \sum_n E_n e^{-\frac{E_n}{kT}} = kT^2 \frac{d}{dT} \ln Z.$

The energy E of the quantum system Q is the expected value of an energy E_n of an energy eigenstate n of the quantum system Q at temperature T.

- **Free Energy** $F = -kT \ln Z.$

The free energy F of the quantum system Q is related to the work performed by the system during a process at constant temperature T.

- **(Statistical Mechanical) Entropy** $S = \frac{E - F}{T}.$

Note that the entropy S of the system Q equals to the Shannon entropy of the probability distribution $\{\mathrm{Prob}(n)\}$, i.e., $S = -k \sum_n \mathrm{Prob}(n) \ln \mathrm{Prob}(n)$.

- **Specific Heat** $C = \frac{dE}{dT}.$

Aim of this talk

We propose a statistical mechanical interpretation of AIT (Algorithmic Information Theory) where $\Omega(D)$ appears as a partition function.

We do this in the following manner:
We introduce the notion of thermodynamic quantities such as free energy, energy, (statistical mechanical) entropy, and specific heat into AIT by performing the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T obeying the canonical distribution:

An energy eigenstate n	$\Longrightarrow$	A program $p \in \mathrm{Dom}\, U$,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p \rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$.

We then determine the convergence or divergence of each of the quantities. In the case where a thermodynamic quantity converges, we calculate the compression rate of the value of the thermodynamic quantity, based on program-size complexity $H(s)$.

$\Longrightarrow$ We see that all of the compression rate of the thermodynamic quantities, which include temperature T itself, equal to T.

Immediate Application of the Replacements: **Transient Definitions**

Perform the following replacements for the corresponding thermodynamic quantities of a quantum system at temperature T. (U: optimal computer)

An energy eigenstate n	$\Longrightarrow$	A program $p \in \operatorname{Dom} U$,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p\rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$. Boltzmann factor: $2^{-\frac{\lvert p\rvert}{T}}$

Partition function $Z(T)=\sum_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad Z(T)=\sum_{p\in \operatorname{Dom} U} 2^{-\frac{\lvert p\rvert}{T}}$,

Free energy $F(T)=-kT\ln Z(T) \quad \Longrightarrow \quad F(T)=-T\log_2 Z(T)$,

Energy $E(T)=\frac{1}{Z(T)}\sum_n E_n e^{-\frac{E_n}{kT}} \quad \Longrightarrow \quad E(T)=\frac{1}{Z(T)}\sum_{p\in \operatorname{Dom} U} \lvert p\rvert\, 2^{-\frac{\lvert p\rvert}{T}}$,

Entropy $S(T)=\frac{E(T)-F(T)}{T} \quad \Longrightarrow \quad S(T)=\frac{E(T)-F(T)}{T}$,

Specific heat $C(T)=\frac{d}{dT}E(T) \quad \Longrightarrow \quad C(T)=\frac{d}{dT}E(T)$.

Thermodynamic Quantities in AIT: **Rigorous Definitions**

Redefine the transient definitions rigorously as follows.

Definition Let $q_1, q_2, q_3, \ldots\ldots$ be an arbitrary enumeration of $\operatorname{Dom} U$. $\square$

Note that the results of this talk are independent of the choice of $\{q_i\}$.

Definition [Thermodynamic Quantities in AIT, Tadaki 2008] Let $T > 0$.

(i) partition function $Z(T) := \lim_{m\to\infty} Z_m(T)$, where $Z_m(T) = \sum_{i=1}^{m} 2^{-\frac{|q_i|}{T}}$.

(ii) free energy $F(T) := \lim_{m\to\infty} F_m(T)$, where $F_m(T) = -T \log_2 Z_m(T)$.

(ii) energy $E(T) := \lim_{m\to\infty} E_m(T)$, where $E_m(T) = \dfrac{1}{Z_m(T)} \sum_{i=1}^{m} |q_i|\, 2^{-\frac{|q_i|}{T}}$.

(iii) entropy $S(T) := \lim_{m\to\infty} S_m(T)$, where $S_m(T) = \dfrac{E_m(T) - F_m(T)}{T}$.

(iv) specific heat $C(T) := \lim_{m\to\infty} C_m(T)$, where $C_m(T) = E'_m(T)$. $\square$

Remark These are variants of Chaitin's Ω. In particular, $Z(T) = \Omega(T)$.

Compression Rate $=$ Temperature.

Thermodynamic Quantities in AIT: **Randomness Property**

Theorem [randomness property, Tadaki, CiE 2008] Let T be a real.
(i) If $0 < T < 1$ and T is computable, then each of $Z(T)$, $F(T)$, $E(T)$, $S(T)$, and $C(T)$ converges to a real whose compression rate equals to T, i.e.,

$$\lim_{n\to\infty} \frac{H(Z(T)\restriction_n)}{n} = \lim_{n\to\infty} \frac{H(F(T)\restriction_n)}{n} = T,$$

$$\lim_{n\to\infty} \frac{H(E(T)\restriction_n)}{n} = \lim_{n\to\infty} \frac{H(S(T)\restriction_n)}{n} = \lim_{n\to\infty} \frac{H(C(T)\restriction_n)}{n} = T.$$

(ii) If $1 < T$, then $Z(T)$, $E(T)$, and $S(T)$ diverge to ∞, and $F(T)$ diverges to $-\infty$.
(iii) In the case of $T = 1$, $C(T)$ diverge to ∞. $\square$

In the case of $T > 1$, it is still open whether $C(T)$ diverges or not.

Implication of (i): The compression rate of the values of all the thermodynamic quantities equals to the temperature T.
Thermodynamic Interpretation of (ii) and (iii): "Phase Transition" occurs at temperature 1.

Remark [Specific Nature of Thermodynamic Quantities in AIT]
The definitions of the thermodynamic quantities in AIT involve the Boltzmann factor $2^{-|p|/T}$. For example, for every $T \in (0,1)$,

$$E(T) = \frac{\sum_{i=1}^{\infty} |q_i|\, 2^{-|q_i|/T}}{\sum_{i=1}^{\infty} 2^{-|q_i|/T}},$$

$$C(T) = \frac{d}{dT}E(T) = \frac{\ln 2}{T^2}\left\{\frac{\sum_{i=1}^{\infty} |q_i|^2\, 2^{-|q_i|/T}}{\sum_{i=1}^{\infty} 2^{-|q_i|/T}} - \left(\frac{\sum_{i=1}^{\infty} |q_i|\, 2^{-|q_i|/T}}{\sum_{i=1}^{\infty} 2^{-|q_i|/T}}\right)^2\right\}.$$

However, note that the compression rate of every function of T involving the Boltzmann factor $2^{-\frac{|p|}{T}}$ does not necessarily equals to T.

To see this, consider the following quantity $\bar{Z}(T)$ which is artificial from the point of view of statistical mechanics:

$$\bar{Z}(T) := \sum_{i=1}^{\infty} \left(2^{-\frac{|q_i|}{T}}\right)^2.$$

Since $\bar{Z}(T) = Z(T/2)$, we see that, for every $T \in (0,1)$, if T is computable then the compression rate of $\bar{Z}(T)$ equals to $T/2$ and not to T.

Temperature $\Rightarrow$ Fixed Point Theorems

In the case where T is computable with $0 < T < 1$,
all of the compression rate of the thermodynamic quantities:

partition function $Z(T)$, free energy $F(T)$,
energy $E(T)$, entropy $S(T)$,and specific heat $C(T)$,

equal to the temperature T.

However,
one of the most typical thermodynamic quantities is temperature T itself.

Thus, the following question arises naturally:

$\boxed{\text{Question}}$ **Can the compression rate of the temperature equal to the temperature itself ?** Self-referential Question

We can answer this question affirmatively in the following form:

Fixed Point Theorem on Compression Rate: **Main Theorem**

Theorem [fixed point theorem on compression rate, Tadaki, CiE 2008]

For every $T \in (0, 1)$, if $Z(T)$ is a computable real, then

$$\lim_{n\to\infty} \frac{H(T\restriction_n)}{n} = T,$$

i.e., the compression rate of T equals to T itself. $\square$

Intuitive Meaning; Metaphor

Consider a file of infinite size whose content is

“The compression rate of this file is $0.100111001 \cdots\cdots$”

When this file is compressed, the compression rate of this file actually equals to $0.100111001 \cdots\cdots$, as the content of this file says.

This situation forms a fixed point and is self-referential !

Remark on the sufficient condition in the fixed Point Theorem

Theorem [fixed point theorem on compression rate] [posted again]

For every $T \in (0,1)$, if $Z(T)$ is computable, then $\lim_{n\to\infty} H(T\restriction_n)/n = T$. $\square$

Note that $Z(T) = \sum_{i=1}^{\infty} 2^{-|q_i|/T}$ is a strictly increasing continuous function of $T \in (0,1)$, and the set of all computable reals is dense in $\mathbb{R}$. Thus,

Theorem The set $\{T \in (0,1) \mid Z(T)$ is computable$\}$ is dense in $(0,1)$. $\square$

Corollary [density of the fixed points]

The set $\{T \in (0,1) \mid \lim_{n\to\infty} H(T\restriction_n)/n = T\}$ is dense in $(0,1)$. $\square$

At this point, the following question would arise naturally:

Question **Is this sufficient condition, i.e., the computability of $Z(T)$, also necessary for T to be a fixed point ?**

Answer **Completely not !! (as we can see through the following argument)**

Thermodynamic Quantities in AIT: **Fixed Point Theorems**

In the fixed point theorem, $Z(T)$ can be replaced by each of the thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$ as follows.

Theorem [fixed point theorem by the free energy $F(T)$, Tadaki, LFCS'09]

For every $T \in (0,1)$, if $F(T)$ is computable, then

$$\lim_{n\to\infty} \frac{H(T\restriction_n)}{n} = T.$$

Theorem [fixed point theorem by the energy $E(T)$, Tadaki, LFCS'09]

For every $T \in (0,1)$, if $E(T)$ is computable, then

$$\lim_{n\to\infty} \frac{H(T\restriction_n)}{n} = T.$$

Theorem [fixed point theorem by the entropy $S(T)$, Tadaki, LFCS'09]

For every $T \in (0,1)$, if $S(T)$ is computable, then

$$\lim_{n\to\infty} \frac{H(T\restriction_n)}{n} = T.$$

These fixed point theorems have the exactly same form as one by $Z(T)$.

Relation between the sufficient conditions of FPTs

Theorem [Tadaki, LFCS'09] There does not exist $T \in (0,1)$ such that both $Z(T)$ and $F(T)$ are computable.

Proof)
Contrarily, assume that both $Z(T)$ and $F(T)$ are computable for some $T \in (0,1)$. Since the statistical mechanical relation $F(T) = -T \log_2 Z(T)$ holds,

$$T = -\frac{F(T)}{\log_2 Z(T)}.$$

Thus, T is computable, and therefore the compression rate of $Z(T)$ equals to T, i.e., $\lim_{n\to\infty} H(Z(T)\restriction_n)/n = T$. This is positive since $T > 0$. On the other hand, since $Z(T)$ is computable by the assumption, the compression rate of $Z(T)$ equals to 0. Thus we have a contradiction. $\square$

$$\underbrace{\{T \in (0,1) \mid Z(T) \text{ is computable}\}}_{\text{dense in } (0,1)} \cap \underbrace{\{T \in (0,1) \mid F(T) \text{ is computable}\}}_{\text{dense in } (0,1)} = \emptyset.$$

In particular, this shows that the computability of $Z(T)$ is not a necessary condition for T to be a fixed point in the fixed point theorem by $Z(T)$.

Relation between the sufficient conditions of FPTs II

Theorem There does not exist $T \in (0,1)$ such that all of $Z(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the statistical mechanical relation

$$S(T) = \frac{E(T)}{T} + \log_2 Z(T).$$

$\square$

Theorem There does not exist $T \in (0,1)$ such that all of $F(T)$, $E(T)$, and $S(T)$ are computable.

Proof) Use the thermodynamic relation

$$S(T) = \frac{E(T) - F(T)}{T}.$$

$\square$

Summary

Temperature = Compression Rate.

Remark: Physical Implication of the Results

Definition Let $q_1, q_2, q_3, \dots\dots$ be an arbitrary enumeration of $\operatorname{Dom} U$. $\square$

In the statistical mechanical interpretation of AIT,
$q_1, q_2, q_3, \dots\dots$ correspond to energy eigenstates of a quantum system and $|q_1|, |q_2|, |q_3|, \dots\dots$ correspond to energy eigenvalues of the quantum system with degeneracy.

Theorem [distribution of programs (i.e., "energy eigenstates"), Solovay]

$$\#\{\, p \mid p \in \operatorname{Dom} U \;\&\; |p| \le n \,\} = 2^{n-H(n)+O(1)} \text{ for all } n \in \mathbb{N}.$$

(In statistical mechanics, this quantity is "the number of states below energy n")

Here $H(n) = H($the base-two representation of $n)$. $\square$

If the energy eigenvalues of a quantum system distribute according to the above distribution, then the following situation can realize:

If T is a computable real, the compression rate of the values of thermodynamic quantities at temperature T equals to T in the quantum system.

Remark: Mathematical Implication of the Results

The proofs of the fixed point theorems on compression rate by $F(T)$, $E(T)$, and $S(T)$ depend heavily on the following thermodynamic relations:

Lemma [thermodynamic relations] $T \in (0, 1)$.

(i) $F'_m(T) = -S_m(T)$, $E'_m(T) = C_m(T)$, and $S'_m(T) = C_m(T)/T$.

(ii) $F'(T) = -S(T)$, $E'(T) = C(T)$, and $S'(T) = C(T)/T$.

(iii) $S_m(T), C_m(T) \geq 0$. $S_m(T), C_m(T) > 0$ for all sufficiently large k.
$S(T), C(T) > 0$. □

Moreover, the proof of the following theorem depends on the statistical mechanical relation $F(T) = -T \log_2 Z(T)$.

Theorem There does not exist $T \in (0, 1)$ such that both $Z(T)$ and $F(T)$ are computable. □

This theorem says that the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$.

These fact would imply that the analytic method can be used in the research of AIT (algorithmic randomness).

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